

Equation of Motion of Small Bodies in Relativity

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Abstract

There is proven a theorem, to the effect that a material body in general relativity, in a certain limit of sufficiently small size and mass, moves along a geodesic.

Within the general theory of relativity, matter is typically described in the following manner. The local state of the matter, at each event of space-time, is characterized by the values there of certain fields on space-time; while the dynamical evolution of that matter is then given by a certain system of partial differential equations on those fields. Now consider a body composed of such matter. Since the system of differential equations on the matter fields normally manifests an initial-value formulation, it follows that every detail of the future behavior of that body is determined, say, from given initial data for that system.

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Fortunately, such a detailed description of the behavior of a material body is in many cases not needed: What is often more interesting is the “behavior of the body as a whole”. Thus, for the case of the Earth, we might be interested only in the Earth’s overall orbit about the Sun, and not in the tides, continental drift, etc. In order to obtain such a description, it is generally necessary to consider a limit of a family of actual material bodies, where that limit involves small overall size of the body, small overall mass of the body, and, possibly, restrictions on the mass-density or other properties of the body. We expect that, on passing to a suitable limit along these lines, the motion of such a body as a whole will be described by a timelike geodesic in space-time. Indeed, the idea that, in some suitable limit, material bodies move on geodesics was an important one already at the beginnings of relativity [1].

It is notoriously difficult even to state a conjecture, in general relativity, reflecting these ideas. For example, neither the “total mass” nor the “size” of a material body arise as natural concepts within the theory, making it awkward to formulate the necessary limitations on the structure of the body. Even more difficult is the problem of how to characterize the motion of the body as a whole. What curve, within the world-tube of the body, will be chosen as “representative” of the body’s overall motion? Will the geodesic character of that curve be with respect to the actual metric of the space-time (which includes, of course, the effects of the body itself), or with respect to some “background metric”? If the latter, how is this background to be defined? Or, alternatively, should the conjecture refer, not to some specific curve, but rather to the “average behavior” of the body?

Despite these difficulties, there have been obtained, in general relativity, a number of results to the effect that material bodies, in a suitable limit, must move on geodesics. For a discussion of various results and approaches to this problem, see, e.g., [2]; and for a summary of later developments see [3], [4].

In one result in particular, [5], there is derived geodesic behavior with respect to a background metric for a body whose gravitational field is ignored, i.e., under the assumption that the background metric remains fixed during passage to the limit of a small body. We shall, essentially, generalize this result to the case in which the body is permitted to manifest a (suitably small) gravitational field of its own, in accordance with the idea outlined above. In more detail, we shall prove the following.

Theorem. *Let M be a 4-manifold, g_{ab} a smooth Lorentz-signature metric on M , and γ a smooth timelike curve on M . Consider a closed neighborhood U of γ and any neighborhood \hat{U} of g_{ab} in $C^1[U]$. Let there exist, for every such U , if sufficiently small, and every such \hat{U} , a Lorentz-signature metric $\tilde{g}_{ab} \in \hat{U}$ whose Einstein tensor i) satisfies the dominant energy condition everywhere in U , ii) is nonzero in some neighborhood of γ , and iii) vanishes on ∂U . Then γ is a g -geodesic.*

Think of U as a “world-tube” surrounding γ . A neighborhood, \hat{U} , of g_{ab} , in the space $C^1[U]$, may be described as follows. Fix, at each point p of U , a neighborhood of $g_{ab}|_p$ (in the space of symmetric tensors at p), and a neighborhood of $\nabla_a|_p$ (in the space of derivative operators at p), where these neighborhoods vary continuously, but otherwise arbitrarily, from point to point in U . Then the metric \tilde{g}_{ab} , for membership in this neighborhood \hat{U} , must be “close to g_{ab} ”, in the following sense: At each point of U , the value of \tilde{g}_{ab} must lie within the given neighborhood of g_{ab} there, and derivative operator, $\tilde{\nabla}_a$, of \tilde{g}_{ab} must lie within the given neighborhood of ∇_a there. Thus, e.g., it follows that, for sufficiently small \hat{U} , the curve γ will again be timelike with respect to every metric \tilde{g}_{ab} in \hat{U} . Note that the metric \tilde{g}_{ab} is defined only within the neighborhood U of the curve γ , and not outside; and that we restrict only the value and first derivative of \tilde{g}_{ab} , and not any higher derivatives. On the Einstein tensor, \tilde{G}_{ab} , of the metric \tilde{g}_{ab} , we impose, in condition i), the following energy condition: $\tilde{G}_{ab}\tilde{t}^a\tilde{t}^b \geq 0$ for any two future-directed \tilde{g} -timelike vectors, \tilde{t}^a and \tilde{t}^b . This implies that, for fixed \tilde{t}^a and \tilde{t}^b , every component of \tilde{G}_{ab} is bounded by a suitable multiple of $\tilde{G}_{ab}\tilde{t}^a\tilde{t}^b$, where that multiple depends, of course, on the frame with respect to which the components of \tilde{G}_{ab} are taken.

Think of the metric \tilde{g}_{ab} as a solution of Einstein’s equation representing a massive body (condition ii)) confined to a neighborhood of γ (condition iii)). The theorem contemplates the existence of a sequence of such solutions, which approach the given “background” metric g_{ab} , in this C^1 -sense. Note that we do *not* require that the stress-energies of the \tilde{g}_{ab} approach the stress-energy of g_{ab} (for that would require C^2 -convergence). Indeed, the stress-energies of the \tilde{g}_{ab} could be unbounded during the approach to g_{ab} . [An example of this behavior is that with M, g_{ab} Minkowski space-time, γ a timelike geodesic therein, and the \tilde{g}_{ab} the metrics of Schwarzschild fluid

balls (of successively smaller radii R), centered on γ and with mass given by $m \propto R^{5/2}$.] Note also that we do not make any assumptions about the stress-energy of g_{ab} itself. In any case, the theorem asserts that, under these conditions, γ is a geodesic with respect to g_{ab} . Note that this implies, in particular, that the \tilde{g} -accelerations of γ approach zero. It is in this sense, then, that the theorem asserts that “small massive bodies move on near-geodesics”.

Proof of the Theorem: Denote by u^a the unit tangent to γ , and set $A^a = u^m \nabla_m u^a$, the acceleration of γ . Let, for contradiction, p_0 be a point of γ at which $A^a \neq 0$.

Choose vector fields t^a , x^a , and β^a , defined in a neighborhood of p_0 , such that: i) at p_0 , $t^a = u^a$, $x^a = A^a/|A^b|$ and $\beta^a = 0$; ii) t^a , x^a and β^a are transported along γ according to the laws $u^m \nabla_m t^a = 0$, $u^m \nabla_m x^a = 0$, and $u^m \nabla_m \beta^a = 2u_b x^{[b} t^{a]}$, respectively; and iii) each of these three vector fields has, everywhere on γ , vanishing symmetrized derivative. [Thus, each of t^a , x^a , and β^a is “Killing on γ ”. Near the point p_0 , t^a and x^a behave like “translations”; while β^a behaves like a “boost”.] Choose points p_+ and p_- of γ , lying on either side of point p_0 , such that

$$\beta^a = f_+ t^a + g x^a, \quad (1)$$

$$\beta^a = f_- t^a - g x^a, \quad (2)$$

at p_+ and p_- , respectively, where $f_+ > 0$, $f_- > 0$ and g are numbers. [That such points exist follows from conditions i) and ii), above, on t^a , x^a , and β^a . Indeed, it follows immediately from these two conditions that, everywhere on γ , β^a is a linear combination of t^a and x^a ; and furthermore that, at p_0 , $\beta^a x_a$ has positive derivative along γ , while $\beta^a t_a$ has zero derivative but negative second derivative.] Note that the nonvanishing of A^a at p_0 was used here, to achieve $f_+ > 0$ and $f_- > 0$. Finally, fix smooth spacelike slices, S_- , S_0 , and S_+ , passing through p_- , p_0 , and p_+ , respectively.

Next, let there be given a neighborhood U of γ in M and neighborhood \hat{U} of g_{ab} in $C^1[U]$, such that, with respect to any metric \tilde{g}_{ab} in \hat{U} , the vector field t^a continues to be timelike in U , and the surfaces S_- , S_0 , and S_+ continue to be spacelike in U . We write “ $\Theta \rightsquigarrow 0$ ” to mean “given any U and \hat{U} as above, the number Θ is bounded; and that bound can be made as small as we wish by choosing U and \hat{U} to be sufficiently small”.

Now let U and \hat{U} be given as above, and let $\tilde{g}_{ab} \in \hat{U}$ satisfy conditions i)-iii) of the Theorem. For S any \tilde{g} -spacelike slice cutting U , and ξ^a any vector field in U , set

$$P(\xi, S) = \int_S \tilde{G}_{ab} \xi^b d\tilde{S}^a, \quad (3)$$

where \tilde{G}_{ab} is the Einstein tensor, and $d\tilde{S}^a$ the surface-element, with respect to \tilde{g}_{ab} . Set $m = P(t, S_0)$. Then, by conditions i) and ii) of the Theorem, $m > 0$. We have, for S and S' any two \tilde{g} -spacelike slices cutting U ,

$$P(\xi, S) - P(\xi, S') = \int_V \tilde{G}_{ab} (\tilde{g}^{c(a} \tilde{\nabla}_c \xi^{b)}) d\tilde{V}, \quad (4)$$

where the integral is over the portion of U between S and S' , and where we have used condition iii) of the Theorem.

This function $P(\cdot, \cdot)$ has three properties of interest.

1. $|P(t, S) - m|/m \rightsquigarrow 0$. This follows from Eqn. (4). Choose $S' = S_0$ and $\xi^a = t^a$ therein, and use the energy condition and the fact that $g^{c(a} \nabla_c t^{b)} = 0$ on γ .

2. If ξ^a vanishes at the point $S \cap \gamma$, then $|P(\xi, S)|/m \rightsquigarrow 0$. This follows from Eqn. (3). By choosing U to be small, we may bound the components of ξ^a on the right; while the energy condition implies that the integral of the components of \tilde{G}_{ab} over S is bounded by a multiple of $P(t, S)$, and so, using property 1, by a multiple of m .

3. If $g^{c(a} \nabla_c \xi^{b)}$ vanishes on γ , then $|P(\xi, S) - P(\xi, S')|/m \rightsquigarrow 0$. This follows from Eqn. (4). By choosing U and \hat{U} to be small, we may bound the components of $\tilde{g}^{c(a} \tilde{\nabla}_c \xi^{b)}$ on the right; while the energy condition and property 1 imply that the integral of the components of \tilde{G}_{ab} over V is bounded by a multiple of m .

Now consider the following number

$$\begin{aligned} K &= P(\beta, S_+) + P(\beta, S_-) - 2P(\beta, S_0) \\ &\quad - f_+ P(t, S_+) - f_- P(t, S_-) - gP(x, S_+) + gP(x, S_-). \end{aligned} \quad (5)$$

We estimate this number K in two ways. For the first, we use property 2 above. Combine the first, fourth, and sixth terms, using (1) and this property; then combine the second, fifth, and seventh, using (2) and this property; and finally apply to the third term this property. We conclude that $|K|/m \rightsquigarrow 0$. For the second way, first combine the first three terms,

using property 3; then combine the last two, again using this property; and finally apply to the two middle terms property 1. We conclude that $|K + (f_+ + f_-)m|/m \rightsquigarrow 0$. But, since the numbers f_+ and f_- are positive, these two estimates contradict each other, completing the proof. \square

We remark that this theorem continues to hold if we weaken the hypothesis to require merely the existence, for each suitable \tilde{g}_{ab} , of some \tilde{g} -conserved symmetric tensor field, \tilde{T}_{ab} , satisfying conditions i)-iii) of the Theorem. This follows, since there was never used in the proof that the \tilde{G}_{ab} in Eqn. (3) is the Einstein tensor of \tilde{g}_{ab} . Thus, for example, the theorem above is also applicable to any metric theory of gravity.

There is a version of the present theorem for Newtonian gravitation. The curve γ is replaced by a fixed time-parameterized curve in space, the background metric g_{ab} by a fixed Newtonian potential ϕ , the \tilde{g}_{ab} by potentials $\tilde{\phi}$, the \tilde{G}_{ab} of Eqn. (3) and its conservation by suitable matter fields and equations, and the conclusion by the assertion that the acceleration of this curve, at each of its points, is given by the value of $-\nabla\phi$ there. Just as it is not necessary (as described in the previous paragraph) to impose Einstein's equation for the present theorem in general relativity, so it is not necessary to impose Poisson's equation for its Newtonian version. This Newtonian result bounds the self-acceleration of a body by the order of the "acceleration of gravity" produced by that body at its surface, e.g., for the Earth, by the order of 10 m/sec². A bound of the same order of magnitude is also available in general relativity, as one sees by following through the proof of the present theorem, keeping track of the inequalities.

In Newtonian gravitation, there is available a much stronger result. (See, e.g., [3], Eqn. (36).) Denote by $\tilde{\phi}$ the actual potential in which a body finds itself, and set ϕ equal to $\tilde{\phi}$ minus the self-potential of that body. Then the difference between the actual acceleration experienced by the center of mass of that body and the "free-fall acceleration" of the center of mass, $-\nabla\phi|_{cm}$, is bounded by the product of the body-size and the variation in the external tidal field (i.e., in $\nabla\nabla\phi$) over that body. For the Earth in its orbit about the Sun, that bound is less than 10⁻¹² m/sec². Unfortunately, it appears to be difficult to find a version of this Newtonian result in general relativity. The key problem, apparently, is that there is in this theory no natural notion of "center of mass" or of "self-potential" [2] [3] [4].

There may be a version of the present theorem applicable to charged par-

ticles, but if there is one it will be somewhat more complicated. For example, the given curve γ and the “background” electromagnetic field together prescribe what is to be the limiting charge-to-mass ratio of the body; and so the hypothesis would have to be modified to require the existence only of bodies having that prescribed ratio. The present theorem is not, of course, applicable to the motion of singular regions or of black holes, for there is no natural “background” with respect to which to describe the behavior of such objects.

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